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1982 J. Phys. A: Math. Gen. 15 L663

(http://iopscience.iop.org/0305-4470/15/12/003)

J. Phys. A: Math. Gen. 15 (1982) L663–L667. Printed in Great Britain

LETTER TO THE EDITOR

Geometry of $N=1$ Yang–Mills theory in curved superspace

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Received 23 September 1982

Abstract. We generalise the formalism of Ogievetsky and Sokatchev to describe the geometry of $N=1$ supersymmetric Yang–Mills theory (coupled in general to supergravity). This generalisation seems to be analogous to the Kaluza theory. Our presentation uses recent developments in geometry of supergravity and exhibits some similarity with the construction of Atiyah and Ward.

The geometrical construction involved in this letter is in fact an application of the general one due to Schwarz (1982) who proposed the general notion of induced geometry. When we consider some surfaces in a space provided with a geometrical structure, this notion gives us the canonical way to define the internal geometry of these surfaces. Being the generalisation of a number of well known mathematical examples, it was shown by Schwarz (1982) to be adequate for the description of $N=1$ supergravity (see also Rosly and Schwarz 1982a, b). In particular, this notion gives the link between the formalism of surfaces and that of 'constrained geometry' in supergravity.

However, we do not refer to the general construction until later. First we show that the supersymmetric Einstein–Yang–Mills system can be described in terms of real surfaces in some complex superspace. Then we present a toy geometrical construction which looks like the construction of Atiyah and Ward for self-dual gauge fields.

The $N=1$ Yang–Mills gauge supermultiplet is known (Ferrara and Zumino 1974, Salam and Strathdee 1974) to be described by the superfield $V=V^it_i$, where (t_i) is a basis of the Lie algebra $\mathscr{K}$ of the group K and V^i are real superfields. The gauge transformation law is

$$e^{2iV}\to e^{\Lambda}\,e^{2iV}\,e^{-\Lambda^*} \tag{1}$$

where $\Lambda=\Lambda^it_i$ and Λ^i are complex chiral superfields. The latter means that $\bar{D}_{\dot\alpha}\Lambda^i=0$, $D_\alpha(\bar{D}_{\dot\alpha})$ being the 'flat' covariant derivatives in superspace (and its complex conjugate). Λ^* in (1) denotes the result of complex conjugation under which t_i are assumed to be inert, i.e. $\Lambda^*=\overline{\Lambda^i}t_i$ (if $\mathrm{K}=\mathrm{U}(N)$ we may take $\Lambda^*=-\Lambda^+$, since then $t_i^+=-t_i$).

In the case when the gauge multiplet V couples to supergravity, we must alter the law (1) in that Λ^i become chiral ($\bar{\mathcal{D}}_{\dot\alpha}\Lambda^i=0$) with respect to 'curved' covariant derivatives. To describe the supergravity itself, Ogievetsky and Sokatchev (1980a, b) proposed an elegant formalism, in which the role of the fields is played by real surfaces of dimension (4, 4) (4 bosonic +4 fermionic) in complex superspace $\mathbb{C}^{4,2}$ (in the minimal formulation). The group $\mathscr{L}$ of all holomorphic transformations of $\mathbb{C}^{4,2}$ which preserve supervolume transforms the surfaces within $\mathbb{C}^{4,2}$. $\mathscr{L}$ is just the group of gauge transformations of the theory.

Now we show that this formalism can be naturally generalised to include the Yang–Mills supermultiplet. The following generalisation proceeds in a manner similar to Kaluza's trick (Kaluza 1921, De Witt 1965). Let us enlarge the space $\mathbb{C}^{4,2}$ by adding an 'internal' space K^c (where K^c is the complexification of the group K). In $\mathbb{C}^{4,2}\times K^c$ consider complex 'coordinates' (x^a, θ^α, k) where $k\in K^c$, $a=1,\ldots,4$; $\alpha=1,2$. Let us describe the group $\hat{\mathscr{L}}$ of gauge transformations which takes the place of $\mathscr{L}$. Besides the transformations of the group $\mathscr{L}$ in the variables (x^a, θ^α) the group $\hat{\mathscr{L}}$ includes also the following holomorphic transformations of $\mathbb{C}^{4,2}\times K^c$:

$$(x^a, \theta^\alpha, k)\to(x^a, \theta^\alpha, \lambda(x,\theta)\cdot k) \tag{2}$$

where $\lambda(x,\theta)\in K^c$. We assume that the real group K is a subgroup of its complexification K^c (if $K=U(N)$ we take $K^c=GL(N,\mathbb{C})$). Denote by π the projection $\pi: K^c\to K^c/K$.

Let us consider real surfaces in $\mathbb{C}^{4,2}\times K^c$ which have dimension equal to $(4+\dim K, 4)$. We restrict ourselves to such surfaces that are invariant under right translations by elements of K. Almost any such surface can be fixed (locally) by the equations

$$\operatorname{Im} x^a = H^a(\operatorname{Re} x, \theta, \bar\theta), \tag{3}$$

$$\pi(k) = F(\operatorname{Re} x, \theta, \bar\theta). \tag{4}$$

If we parametrise this surface by $(\xi^a, \nu^\alpha, \bar\nu^{\dot\alpha}, a)$ where ξ^a are real (even), ν^α, $\bar\nu^{\dot\alpha}$ are complex (odd) parameters and $a\in K$, we get

$$x^a=\xi^a+\mathrm{i}H^a(\xi,\nu,\bar\nu), \qquad \theta^\alpha=\nu^\alpha, \qquad k=\mathrm{e}^{\mathrm{i}V(\xi,\nu,\bar\nu)}\cdot a. \tag{5}$$

Here $\mathrm{e}^{\mathrm{i}V}$ takes the place of F in (4) with $\mathrm{i}V(\xi,\nu,\bar\nu)$ lying in the 'purely imaginary' part of the Lie algebra of K^c, so $V(\xi,\nu,\bar\nu)$ can be considered as taking values in $\mathscr{K}$, the Lie algebra of K. It is the superfield H^a (cf (5)) which contains the component fields of supergravity with the correct transformation properties under the group $\mathscr{L}$ (Ogievetsky and Sokatchev (1980a, b). Let us consider the action of the remaining part of $\hat{\mathscr{L}}$ given by (2). It can be easily seen that V transforms as in (1), with $\mathrm{e}^\Lambda=\lambda(\xi+\mathrm{i}H,\nu)$. The function Λ takes values in the complexification of $\mathscr{K}$ and the coordinates Λ^i are chiral fields with respect to 'curved' derivatives (proportional to $\partial/\partial\nu^\alpha+\mathrm{i}H^a_{,\alpha}\partial/\partial\xi^a$). Thus the fields in (5) (determining the real surface in $\mathbb{C}^{4,2}\times K^c$) indeed correspond to the supersymmetric Einstein–Yang–Mills system. The restriction (4) of the form of the surfaces (their invariance under the right action of K) becomes natural from the more general geometrical point of view, to be discussed later.

The gauge supermultiplet can be alternatively described by means of the constrained gauge potential $\mathscr{A}_A$ in superspace (see e.g. Sohnius 1978). The constraints on its field strengths (in the tangent frame D_a, D_α, $\bar D_{\dot\alpha}$) are

$$F_{\alpha\beta}=0, \qquad F_{\dot\alpha\dot\beta}=0, \tag{6}$$

and one more constraint $F_{\alpha\dot\beta}=0$. The last constraint, however, serves only to determine the components $\mathscr{A}_a$ in terms of $\mathscr{A}_\alpha$. We do not consider $\mathscr{A}_a$ at all. The geometrical object under consideration is only $\mathscr{A}_\alpha$ constrained by (6). We note that the constraint

$F_{\alpha\beta}=0$ is simply the gauge condition in an appropriate formalism. The gauge superfield V discussed above arises as the solution to (6).

Let us consider this geometry in more detail. In the superspace $\mathcal{M}$, usually considered in supergravity, the tangent space at each point is spanned by complex vectors D_α ($\bar{D}_{\dot\alpha}$) and real ones D_a. In other words, there is a distinguished tangent subspace $H_z \subset T_z(\mathcal{M})$ equipped with the complex structure (i.e. multiplication by the imaginary unit is defined in H_z) at each point $z \in \mathcal{M}$). The structure of the above type in any manifold is generally called the Cauchy–Riemann structure (CR-structure). For some given CR-structure in $\mathcal{M}$ consider complex vector fields D_α which span the complex basis of H_z at each point. Let D_a be some real vector fields which extend D_α, $\bar{D}_{\dot\alpha}$ to a basis $D_A=(D_a, D_\alpha, \bar{D}_{\dot\alpha})$ of the whole tangent space at every point. In general, we have $[D_\alpha, D_\beta\}=C^B_{\alpha\beta}D_B$, where $[\,,\}$ denotes the Lie bracket in a supermanifold, i.e. (anti)commutator of vector fields. The CR-structure is called integrable if the following constraints are satisfied: $C^b_{\alpha\beta}=0$, $C^{\dot\gamma}_{\alpha\beta}=0$. This is just the case of $N=1$ supergravity, due to the constraints imposed on the torsion: $T^b_{\alpha\beta}=0$, $T^{\dot\gamma}_{\alpha\beta}=0$ (see Schwarz 1982).

The above super Yang–Mills gauge potential ($\mathcal{A}_\alpha$, $\mathcal{A}_{\dot\alpha}$) is a one-form defined on vectors belonging to the tangent subspace H_z with values in the Lie algebra $\mathcal{K}$ of the gauge group K. More geometrically, we have the principal bundle P with the structure group K and the way to lift (horizontally) the vectors of H_z (by means of ($\mathcal{A}_\alpha$, $\mathcal{A}_{\dot\alpha}$)). Let us pass now to the bundle P^c with the structure group K^c (the complexification of K) and prolong the 'semiconnection' ($\mathcal{A}_\alpha$, $\mathcal{A}_{\dot\alpha}$) on the whole P^c. We can provide P^c with a CR-structure using the complex structure of fibres K^c and the lifts of H_z's from the base. Now the constraints (6) on ($\mathcal{A}_\alpha$, $\mathcal{A}_{\dot\alpha}$) mean that this CR-structure is integrable, provided the CR-structure in the base $\mathcal{M}$ is integrable. Thus we obtain a bundle with the complex structure group K^c and this bundle has transition functions $\varphi(z)$ which are chiral, i.e. satisfy the Cauchy–Riemann equations, $\bar{D}_{\dot\alpha}\varphi=0$, in a given CR-structure of the base manifold $\mathcal{M}$. Such bundles could be called integrable CR-bundles, or chiral bundles (note that one could equally well understand the abbreviation CR either as 'Cauchy–Riemann' or 'chiral'). The above correspondence is one-to-one. In fact we have a simple analogue of the Atiyah–Ward (1977) construction.

Provided an integrable CR-structure in $\mathcal{M}$ is given, there is a one-to-one correspondence between the super Yang–Mills gauge potentials in $\mathcal{M}$ ('semiconnections' ($\mathcal{A}_\alpha$, $\mathcal{A}_{\dot\alpha}$) defined along vectors D_α, $\bar{D}_{\dot\alpha}$ in $\mathcal{M}$) satisfying constraints (6), on the one hand, and chiral bundles over $\mathcal{M}$ with structure group K^c and with a reduction to the real subgroup K, on the other hand. The above-mentioned reductions are determined as usual by sections of the associated bundle with K^c/K as a fibre. Such a section can be given (locally) by the function $\exp \mathrm{i}V(\xi, \nu, \bar\nu)$, where $(\xi, \nu, \bar\nu)$ are coordinates in superspace $\mathcal{M}$ and $V(\xi, \nu, \bar\nu)\in\mathcal{K}'$ is just the gauge supermultiplet mentioned earlier. In the case of coupling to supergravity the CR-structure in the base $\mathcal{M}$ is determined by the field of supergravity, as already pointed out (see also below).

An example where CR-structures are relevant is the geometry of real surfaces in some complex space $\mathcal{N}$. In this case the subspace H_z in the tangent space $T_z(\mathcal{M})$ to a surface $\mathcal{M}$ at $z\in\mathcal{M}$ is defined as the so-called maximal complex tangent subspace. This means that if we consider $T_z(\mathcal{M})$ as the subspace in $T_z(\mathcal{N})$, we can multiply the vectors of $T_z(\mathcal{M})$ by the imaginary unit (because of the complex structure in $\mathcal{N}$). Let us define $H_z=T_z(\mathcal{M})\cap J(T_z(\mathcal{M}))$, where J denotes the operator of multiplication by the imaginary unit. We see that $\{H_z, z\in\mathcal{M}\}$ define a CR-structure in $\mathcal{M}\subset\mathcal{N}$ induced by the complex structure of the space $\mathcal{N}$. It can easily be seen that such CR-structures are always integrable. The inverse will also be true if all the objects under consideration are supposed to be real analytic. We will later assume this about surfaces, potentials, etc (this assumption is, however, not strictly necessary).

As was pointed out above, in supergravity we deal with integrable CR-structures in a superspace of real dimension (4, 4). The formalism of Ogievetsky and Sokatchev gives the realisation of these structures on real surfaces in $\mathbb{C}^{4,2}$ (see Schwarz 1982). In view of the correspondence between super Yang–Mills potentials and integrable CR-bundles we can mimic the formalism of surfaces. In fact, this was done above. As the CR-structure of the base is realised on a surface $\mathcal{M}$ in $\mathbb{C}^{4,2}$, the constraints (6) on the super Yang–Mills gauge potential mean that the corresponding CR-bundle can be uniquely prolonged to a holomorphic bundle in a neighbourhood of the surface $\mathcal{M}$ in $\mathbb{C}^{4,2}$. For example, the CR-structure of the 'flat' superspace is known to be realised on the quadric Q in $\mathbb{C}^{4,2}$ defined by the equations $\operatorname{Im} x^a = 2\mathrm{i}\sigma^a_{\alpha\dot\beta}\theta^\alpha\bar\theta^{\dot\beta}$ (x^a, θ^α are complex coordinates in $\mathbb{C}^{4,2}$ and σ are the Pauli matrices). Then the super Yang–Mills potentials in 'flat' superspace correspond to holomorphic bundles in neighbourhoods of Q in $\mathbb{C}^{4,2}$ with reductions to the real subgroup and *vice versa*.

Let us pass now to a more general point of view using the notion of induced geometry (see Schwarz 1982). Here the word 'geometry' means that the so-called G-structures are understood (see e.g. Sternberg 1964). G-structure is a tool to describe geometry in terms of vector bases (frames) at each point. Let G be a matrix group acting in the tangent space of the manifold $\mathcal{M}$. One says that two frames (at some point) (e_a) and $(\tilde e_a)$, $a = 1, \ldots, \dim\mathcal{M}$ are G-equivalent when $\tilde e_a = g^b_a e_b$, for some matrix $(g^b_a) \in \mathrm{G}$. If the sets of G-equivalent frames in all points of $\mathcal{M}$ are fixed, one says that the G-structure is given in $\mathcal{M}$. (In general relativity, for example, it is useful to consider tetrad fields, i.e. L-structures with L being the Lorentz group.)

Consider some space $\mathcal{N}$ with a G-structure. If $\dim\mathcal{N} = n$, then $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$. If $\mathcal{M} \subset \mathcal{N}$ is an m-dimensional surface, we can try to provide it with some G′-structure. Let $(e_{\hat a})$, $\hat a = 1, \ldots, n$ be a frame at $z \in \mathcal{M}$ such that $(e_{\hat a})$ belongs to the given G-structure in $\mathcal{N}$ and its first m vectors e_a, $a = 1, \ldots, m$, are tangent to $\mathcal{M}$. It is not always possible to choose such a frame (called an adopted frame). If such frames can actually be chosen at all points of $\mathcal{M}$, we say that such a surface $\mathcal{M}$ in $\mathcal{N}$ is regular. All adopted frames at a point are connected by the transformations with matrices in the subgroup $\tilde{\mathrm{G}}$ of the group G. $\tilde{\mathrm{G}}$ consists of the matrices $(g^{\hat b}_{\hat a})$ of the group G which satisfy $g^{b'}_a = 0$ if $a = 1, \ldots, m$; $b' = m + 1, \ldots, n$. The first m vectors of an adopted frame, being tangent to $\mathcal{M}$, also define a frame in $\mathcal{M}$. Such frames in $\mathcal{M}$ are defined up to the transformations of the group $\mathrm{G}' \subset \mathrm{GL}(m, \mathbb{R})$ which is a certain factor group of $\tilde{\mathrm{G}}$. Thus we see that every regular surface in $\mathcal{N}$ carries a G′-structure induced by the G-structure of $\mathcal{N}$ (Schwarz 1982).

The CR-structure on the surface in the complex space presents an example of the induced structure (for details see Schwarz 1982). This example shows, in particular, that the induced geometry is not an arbitrary one. Indeed, induced CR-structures are necessarily integrable (see above). In general, G′-structure on a surface in a space carrying some G-structure satisfies certain conditions. These conditions are imposed on the so-called structure functions of the G′-structure. For details and for discussion of the sufficiency of these conditions see Schwarz (1982) and Rosly and Schwarz (1982a, b). It is revealed there that these conditions give the torsion and curvature constraints in $N = 1$ supergravity.

One more example of the induced structure was implicitly treated above: there we can take the space $\mathcal{N}$ carrying the structure of the holomorphic principal K^c-bundle over the base $\mathbb{C}^{4,2}$ with the fixed volume element in $\mathbb{C}^{4,2}$. Let us consider real

$(4+\dim K, 4)$-dimensional surfaces in $\mathcal{N}$. It can be easily seen that the surfaces (3), (4) are just regular surfaces in $\mathcal{N}$. One can find that the geometry, induced on such a surface, corresponds to the 'semiconnection' (in presence of supergravity). Now the general 'torsion constraints' of induced geometry give the constraints (6) (and the supergravity torsion constraints). This provides the link between the gauge superfield V, which emerges in the description of regular surfaces, and the superpotential $(\mathcal{A}_\alpha, \mathcal{A}_{\dot\alpha})$ which defines the 'semiconnection'.

Let us illustrate some details of the above considerations by a simple example. As is well known, there exists the canonical K-connection in the holomorphic K^c-bundle with a given reduction to the real subgroup $K \subset K^c$. (In the case $K = U(N)$, $K^c = GL(N, \mathbb{C})$, this is the canonical connection in the holomorphic Hermitian bundle.) Consider some local trivialisation of the reduced K-bundle and take z^α as a holomorphic coordinate in the base. Then the canonical connection in the reduced bundle is specified by the constraints on its field strength (curvature) $F_{\alpha\beta} = 0$, $F_{\dot\alpha\dot\beta} = 0$. Let us show that this construction corresponds to an induced geometry.

The structure of the holomorphic principal K^c-bundle can be described by the following G-structure in its total space P^c. At each point of P^c, there are complex vectors t_i and $\partial/\partial z^\alpha$, where t_i correspond to some fixed complex basis of the Lie algebra of K^c. Then the G-structure in P^c is described by the frames consisting of real and imaginary parts of the vectors t_i, $\partial/\partial z^\alpha$. Consider the surfaces in P^c which are tangent at each point to vectors $\mathrm{Re}\, t_i$, $\partial/\partial z^\alpha$ (in coordinates z^α chosen appropriately for each point of a surface). These surfaces are regular. On the other hand, we see that any such surface $P \subset P^c$ is a reduction of P^c to the real subgroup $K \subset K^c$. The induced G'-structure is described by the frames $(\mathrm{Re}\, t_i, \mathrm{Re}\, \mathcal{D}_\alpha, \mathrm{Im}\, \mathcal{D}_\alpha)$, where $\mathcal{D}_\alpha$ are some complex vectors (of type $(1, 0)$). The group G' turns out to consist of the following transformations of these frames: $t_i \to t_i$, $\mathcal{D}_\alpha \to C^\beta_\alpha \mathcal{D}_\beta$. As a result we obtain the tangent (horizontal) subspace H_p, spanned by $\mathrm{Re}\, \mathcal{D}_\alpha$, $\mathrm{Im}\, \mathcal{D}_\alpha$, at each $p \in P$. The general 'torsion constraints' of the induced geometry show in this case that H_p's are invariant under the right action of K and thus the defined connection in P satisfies $F_{\alpha\beta} = 0$, $F_{\dot\alpha\dot\beta} = 0$. We conclude that the induced geometry in $P \subset P^c$ corresponds to the canonical connection in the reduced bundle.

I am grateful to Professor A S Schwarz for introducing me to induced structures and also for valuable discussions. I thank A A Tseytlin for helpful discussions and remarks. I thank also E A Ivanov, who pointed out that some of the results of this letter intersect with his unpublished results.

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